Journal of Statistical Physics, Vol. 122, No. 1, January 2006 (© 2006)
DOI: 10.1007/s10955-005-8084-9

Remarks on the Balance Relations for the Two-Dimensional Navier–Stokes Equation with Random Forcing

Sergei B. Kuksin[1]

Received April 25, 2005; accepted September 14, 2005

We use the balance relations for the stationary in time solutions of the randomly forced 2D Navier-Stokes equations, found in [10], to study these solutions further. We show that the vorticity $\xi(t, x)$ of a stationary solution has a finite exponential moment, and that for any $a \in \mathbb{R}$, $t \geq 0$ the expectation of the integral of $|\nabla_x \xi|$ over the level-set $\{x \mid \xi(t, x) = a\}$, up to a constant factor equals the expectation of the integral of $|\nabla_x \xi|^{-1}$ over the same set.

KEY WORDS: Two-dimensional Navier–Stokes equation; stationary measure; vorticity; balance relations; exponential moment.

1. INTRODUCTION

In this paper we continue to study the 2D Navier–Stokes equation on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$, perturbed by a random force:

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p(t, x) = \sqrt{\nu}\, \tilde{\eta}(t, x),$$
$$u = u(t, x) \in \mathbb{R}^2, \quad x \in \mathbb{T}^2, \quad \operatorname{div} u = 0, \quad \int u\, dx \equiv \int \tilde{\eta}\, dx \equiv 0. \tag{1.1}$$

The scaling factor $\sqrt{\nu}$ in the r.h.s. of the equation is not important for us (since the force $\tilde{\eta}$ may depend on ν), but it makes the formulas obtained nicer.

The force $\tilde{\eta}$ is a stationary random field in t and x, smooth periodic in x and white in t. If $\tilde{\eta}$ satisfies some mild nondegeneracy assumptions, discussed in the next section, then the probability distribution of any solution $u(t, x)$ of the stochastic differential equation (1.1) converges as $t \to \infty$ to a unique stationary measure (which is a measure on a function space, forms by divergence–free vector

[1] Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS Scotland, UK, and Steklov Institute of Mathematics, 8 Gubkina St., 117966 Moscow, Russia; e-mail: kuksin@ma.hw.ac.uk

fields $u(x)$). Let $u_\nu(t,x)$, $t \geq 0$, be a stationary in time solution of (1.1), for any t distributed as the stationary measure, and let $\xi_\nu(t,x)$ be its vorticity:

$$\xi_\nu(t,x) = \operatorname{curl} u_\nu = \frac{\partial u_\nu^2}{\partial x_1} - \frac{\partial u_\nu^1}{\partial x_2}\,.$$

The vorticity ξ_ν satisfies the diffusion–convection equation

$$\dot{\xi} - \nu\Delta\xi + (u\cdot\nabla)\xi = \sqrt{\nu}\,\operatorname{curl}\tilde{\eta}. \tag{1.2}$$

Since $\tilde{\eta}$ is stationary in x, then both u_ν and ξ_ν are stationary in x as well.

It is established in [10] that the process ξ_ν satisfies infinitely many *balance relations*:

$$\mathbf{E}(g(\xi_\nu(t,x))\mid\nabla\xi_\nu(t,x)|^2) = B_1\mathbf{E}(g(\xi_\nu(t,x)) \quad \forall\, t,x\,. \tag{1.3}$$

Here g is any continuous function which has at most a polynomial growth, i.e. $|g(v)| \leq C(1+|v|^k)$ for all v, where C and k are some fixed constants. The constant B_1 is explicitly defined in terms of the force $\tilde{\eta}$. The relations (1.3) are related to the Helmholtz invariants for inviscid 2d flow.

The goal of this work is to derive from the balance relations two corollaries. The first corollary[2] deals with integrals over level–sets $\Gamma(\tau)$ of the vorticity ξ_ν,

$$\Gamma(\tau) = \{x \mid \xi_\nu(t,x) = \tau\},$$

where $t \geq 0$ is fixed. Each $\Gamma(\tau)$ is a random subset of the torus $\mathbb{T}^2$ (the notation of a random parameter is suppressed everywhere in the Introduction). In Theorem (3.2), Section (3), we derive the following *co–area form of the balance relations*:

$$\mathbf{E}\int_{\Gamma(\tau)} |\nabla\xi_\nu|\,d\gamma = B_1\mathbf{E}\int_{\Gamma(\tau)} |\nabla\xi_\nu|^{-1}\,d\gamma\,, \tag{1.4}$$

for almost all τ. Here $d\gamma$ is the length element on the random curve $\Gamma(\tau)$, which is well defined a.s.

If τ is a regular value of ξ_ν as a function of x, then $\Gamma(\tau)$ is a finite union of smooth curves, and in any point $x \in \Gamma(\tau)$ we have $\nabla\xi_\nu = \pm\frac{\partial}{\partial n}\xi_\nu$, where n is a unit normal to the curve. Hence, in (1.4) $|\nabla\xi_\nu|$ can be replaced by $|\frac{\partial}{\partial n}\xi_\nu|$. So the integral $\int |\nabla\xi_\nu|\,d\gamma$ is a sum of the moduli of the flows of $\nabla\xi_\nu$ through the curves, forming the set Γ. The integral $\int |\nabla\xi_\nu|^{-1}\,d\gamma$ can be interpreted similar.

If $u(t,x)$ is any solution of (1.1) and $\xi(t,x)$ is its vorticity, then by the ergodic theorem the average in ensemble can be replaced by the average in time. Therefore, $\xi(t,x)$ satisfies (1.4), where the expectation $\mathbf{E}$ is replaced by $\lim T^{-1}\int_1^{T+1}\dots\,dt$ (cf. the end of Section (2), where the balance relations (1.3) are re–interpreted similar).

[2] In fact, this is an equivalent reformulation of the relations.

The level sets $\Gamma(\tau)$ of the vorticity of a solution for the *deterministic* Navier–Stokes equation in the 2d and 3d cases, and of solutions for equation (1.2) without assuming that $\xi = \operatorname{curl} u$ (but imposing certain a-priori bounds on u and ξ), were studied by P. Constantin and others in [2, 4, 5]. There the areas of the sets $\Gamma(\tau)$ are estimated (with and without averaging in t and τ), as well as certain integrals over these sets. The results of [2, 5] are physically motivated. At this moment we cannot suggest any physical interpretation of the relations (1.4). We simply believe that they are important as exact relations, satisfied by solutions of a basic equation of mathematical physics.

In Section (4) the balance relations are used to prove that the random field ξ_ν has finite exponential moments:

$$\mathbf{E}e^{\sigma|\xi_\nu(t,x)|} \le C < \infty$$

for some $\sigma > 0$, uniformly in t, x and in $\nu > 0$. Moreover, $\mathbf{E}e^{\sigma_1|u_\nu(t,x)|} \le C_1$ and $\mathbf{E}e^{\sigma_2|\nabla_x u_\nu(t,x)|^{1/2}} \le C_2$ (for all t, x, uniformly in $\nu > 0$). In particular,

$$\mathbf{P}\{|\xi_\nu(t,x)| \ge K\} \le Ce^{-\sigma K} \quad \forall K ,$$

etc. That is, high values of $u_\nu(t,x)$, or of $\xi_\nu(t,x)$, or of $\nabla u_\nu(t,x)$ are very unlikely.

2. PRELIMINARIES

We denote by H the space of square–integrable vector fields $u(x)$ such that $\operatorname{div} u = 0$ and $\int u\,dx = 0$ given the L_2-norm $\|\cdot\|$ and the L_2-scalar product $(\cdot,\cdot)$. By H^n, $n \in \mathbb{N}$, we denote the Sobolev space $H^n = H \cap H^n(\mathbb{T}^2;\mathbb{R}^2)$ given the norm $\|u\|_n = \big((-\Delta)^n u, u\big)^{1/2}$.

Let us denote by Π the Leray projector $\Pi : L_2(\mathbb{T}^2;\mathbb{R}^2) \to H$, which removes the gradient and the constant part of a vector field it operates upon (see e.g., [3]). Applying Π to (1.1) we write it in the usual form

$$\dot u(t,x) + \nu L u + B(u) = \sqrt{\nu}\,\eta, \tag{2.1}$$

where we have denoted $Lu = -\Pi\Delta u$, $B(u) = \Pi(u\cdot\nabla)u$ and $\eta = \Pi\tilde\eta$.

The force $\eta(t,x)$ is assumed to be a Gaussian random field, white in time and smooth in x:

$$\eta = \frac{d}{dt}\zeta(t,x), \quad \zeta = \sum_{s\in\mathbb{Z}_0^2} b_s\beta_s(t)e_s(x). \tag{2.2}$$

Here $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$, $\{\beta_s(t) = \beta_s^\omega(t)\}$ is a set of independent standard Wiener processes, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, satisfying $\beta_s(0) = 0$.

The real coefficients b_s are such that $M_1 < \infty$, where

$$M_n = \sum_{s \in \mathbb{Z}_0^2} |s|^{2n} b_s^2, \quad n \in \mathbb{Z},$$

and $(e_s, s \in \mathbb{Z}_0^2)$ is the standard trigonometric basis of H:

$$e_s(x) = \frac{\sin(s \cdot x)}{\sqrt{2}\pi |s|} \begin{bmatrix} -s_2 \\ s_1 \end{bmatrix} \quad \text{if } s_1 + s_2 \delta_{s_1,0} > 0$$

$$e_s(x) = \frac{\cos(s \cdot x)}{\sqrt{2}\pi |s|} \begin{bmatrix} -s_2 \\ s_1 \end{bmatrix} \quad \text{if } s_1 + s_2 \delta_{s_1,0} < 0.$$

The vector fields e_s are eigenvectors of L: $Le_s = |s|^2 e_s$ for each s.

We shall often interpret random fields $u(t,x)$ and $\eta(t,x)$ as random processes in H (or in another space H^n) and write them as $u(t)$ and $\eta(t)$.

It is well known (see [6, 8, 18]) that for any random initial data u_0, independent of $\eta(\cdot)$, and such that $\mathbf{E}\|u_0\|^2 < \infty$, the equation (2.1) has a unique solution $u(t)$, belonging to $C([0,\infty), H)$ almost sure. This solution is a Markov process in H. We shall denote it as $u = u(t; u_0)$. If

$$b_s \neq 0 \quad \forall s \tag{2.3}$$

then the Markov process has a unique stationary measure μ_ν, [3] and all solutions $u(t)$ as above converge to μ_ν in distribution:

$$\mathcal{D}u(t) \rightharpoonup \mu_\nu \quad \text{as} \quad t \to \infty.$$

Here and below $\mathcal{D}$ signifies the distribution of a random variable. If we choose for u_0 a random vector in H, distributed as μ_ν, then the corresponding solution, which will be denoted as $u_\nu(t)$, $t \geq 0$, is stationary:

$$\mathcal{D}(u_\nu(t)) = \mu_\nu \quad \forall t \geq 0.$$

Now let us assume that, in addition to (2.3), the coefficients b_s are symmetric in s:

$$b_s = b_{-s} \neq 0 \quad \forall s. \tag{2.4}$$

Then the random field $\zeta(t,x)$ is translationary invariant. That is, its distribution is invariant under the translation T_h of the torus $\mathbb{T}^2$, $T_h x = x + h$ $(h \in \mathbb{T}^2)$. Due to the uniqueness, this implies that the stationary measure μ_ν and the stationary

[3] See [1, 7, 11, 12] and see [14] for discussion of this result and its development. Recently it was announced in [9] that the stationary measure is unique if (2.3) holds for all $|s| \leq 2$ (previously it was known that the result is true if (2.3) holds for $|s| \leq N$, where $N = N(\nu) < \infty$ is sufficiently large).

process $u_\nu(t,x)$ also are translationary invariant:

$$T_h\mu_\nu = \mu_\nu, \quad \mathcal{D}u_\nu(\cdot, \cdot + h) = \mathcal{D}u_\nu(\cdot,\cdot) \quad \forall\, h \in \mathbb{T}^2,$$

see [10, 14]. Under assumptions (2.4) the vorticity $\xi_\nu = \operatorname{curl} u_\nu$ satisfies the balance relations (1.3), where $B_1 = 1/2(2\pi)^{-2}M_1$, see [10].

It is known (e.g., see [18, 8, 13]) that if $M_k < \infty$ $(k \ge 1)$, and $u_0 \in H$ is a non-random vector, then for any $T \ge 2$ the solution $u = u(t; u_0)$ satisfies

$$u \in C([1,T], H^k) \quad \text{and} \quad u - \sqrt{\nu}\,\zeta \in C^1([1,T], H^{k-2}) \quad \text{a.s.}. \tag{2.5}$$

The stationary solutions $u_\nu(t,x)$ also satisfy (2.5). Moreover, in this case the relations hold with $[1,T]$ replaced by any finite segment $[T_1,T_2]$, $T_1 \ge 0$.

For $u_0 \in H$, let us denote $\xi(t,x;u_0) = \operatorname{curl} u(t,x;u_0)$. Noting that for any $x \in \mathbb{T}^2$ the map $H^4 \to \mathbb{R}^2$, $u(\cdot) \mapsto (\operatorname{curl} u(x), |\nabla \operatorname{curl} u(x)|^2)$, is continuous, we see that

$$\frac{1}{T}\int_1^{T+1} g(\xi(t,x;u_0))|\nabla\xi(t,x;u_0)|^2\,dt \to \mathbf{E}(g(\xi_\nu(x)|\nabla\xi_\nu(x)|^2) \quad \text{a.s.},$$

by the Strong Law of Large Numbers (see [14, 16]), if

$$M_5 < \infty, \quad \mathbf{E}|u_0|^2 < \infty, \quad g(\xi) \text{ is a polynomial.} \tag{2.6}$$

Similar result holds for the functional $\xi \to g(\xi(t,x))$, so *the balance relation (1.3) still holds if we replace* $\mathbf{E}$ *by* $\lim T^{-1}\int_1^{T+1}\dots dt$ *and replace* ξ_ν *by any* $\xi(t,x;u_0)$*, provided that (2.6) is satisfied.*

3. THE CO-AREA FORM OF THE BALANCE RELATIONS

From now on we assume that

$$M_6 < \infty.$$

Then, by (2.5), there exists a null-set Ω_0 such that for $\omega \notin \Omega_0$ we have

$$\xi_\nu \in C([0,T]; C^3(\mathbb{T}^2)), \quad \xi_\nu - \sqrt{\nu}\operatorname{curl}\zeta \in C^1([0,T]\times\mathbb{T}^2). \tag{3.1}$$

We re-define ξ_ν and ζ to vanish for $\omega \in \Omega_0$. Now (3.1) holds for all ω.

Lemma 3.1. *For any t, x and ν we have*

$$\mathbf{P}\{\nabla\xi_\nu(t,x) = 0\} = 0. \tag{3.2}$$

The lemma is proved in Appendix.

Let us fix any $t \ge 0$, abbreviate $\xi_\nu(t,x) = \xi(x)$ and integrate the balance relation over $\mathbb{T}^2$:

$$\mathbf{E}\int_{\mathbb{T}^2} g(\xi(x)|\nabla\xi(x)|^2\,dx = B_1\mathbf{E}\int_{\mathbb{T}^2} g(\xi(x))\,dx. \tag{3.3}$$

For a continuous function g such that $|g| \leq 1$, we denote

$$I_1 = \mathbf{E} \int_{\mathbb{T}^2} g(\xi(x))|\nabla\xi(x)|^2 \, dx \,, \quad I_2 = \mathbf{E} \int_{\mathbb{T}^2} g(\xi(x)) \, dx.$$

Then

$$I_1 = B_1 I_2. \tag{3.4}$$

For $\varepsilon > 0$ let us consider the set $K_\varepsilon \subset \mathbb{T}^2 \times \Omega$,

$$K_\varepsilon = \{(x, \omega) \mid |\nabla\xi| \geq \varepsilon\}.$$

Clearly,

$$I_1 = I_1^\varepsilon + O(\varepsilon^2), \quad I_1^\varepsilon = \mathbf{E} \int_{\mathbb{T}^2} g(\xi(x)|\nabla\xi(x)|^2 I_{K_\varepsilon}(x, \omega) \, dx.$$

For the quantity I_2^ε, obtained from I_2 by multiplying the integrand by the factor I_{K_ε}, we have

$$|I_2 - I_2^\varepsilon| \leq \int_\Omega \int_{T^2} I_{\{|\nabla\xi|<\varepsilon\}}(x, \omega) \, dx \, P(d\omega).$$

Since the integrand converges to $I_{\{\nabla\xi=0\}}$ when $\varepsilon \to 0$, then by Lemma (3.1) we have

$$I_2^\varepsilon \to I_2 \quad \text{as} \quad \varepsilon \to 0 \,.$$

Let us denote $K_\varepsilon(\omega) = \{x \mid (x, \omega) \in K_\varepsilon\}$. Then by the implicit function theorem, for every ω, every $\tau \in \mathbb{R}$ and any $\varepsilon > 0$, the set

$$\Gamma_\varepsilon(\tau, \omega) = \{x \in K_\varepsilon(\omega) \mid \xi(x) = \tau\}$$

is a finite union of C^3-smooth curves of finite length. We shall denote by γ points of a set $\Gamma_\varepsilon(\tau, \omega)$ and denote by $d\gamma$ the length-element. Performing the 'co-area change of variables'

$$K_\varepsilon(\omega) \ni x \to (\tau, \gamma), \quad \tau = \xi(x), \ \gamma \in \Gamma_\varepsilon(\tau, \omega),$$

we have $d\tau \, d\gamma = |\nabla\xi| \, dx_1 \, dx_2$. So

$$I_1^\varepsilon = \mathbf{E} \int_{\mathbb{R}} g(\tau) \int_{\Gamma_e(\tau,\omega)} |\nabla\xi| \, d\gamma \, d\tau \,,$$

and

$$I_2^\varepsilon = \mathbf{E} \int_{\mathbb{R}} g(\tau) \int_{\Gamma_\varepsilon(\tau,\omega)} |\nabla\xi|^{-1} \, d\gamma \, d\tau \,.$$

Since the map ξ is C^3-smooth, then the Sard lemma applies and for every ω and almost every $\tau \in \mathbb{R}$ the level-set

$$\Gamma(\tau, \omega) = \{x \in \mathbb{T}^2 \mid \xi(x) = \tau\} \tag{3.5}$$

is a C^3-smooth manifold. So

$$\int_{\Gamma_e(\tau,\omega)} |\nabla\xi|\, d\gamma \nearrow \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma < \infty \ \text{ as } \ \varepsilon \to 0\,, \tag{3.6}$$

for every ω and a.e. τ. Hence,

$$I_1^\varepsilon \to \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma\, d\tau \ \text{ as } \ \varepsilon \to 0\,,$$

by the monotone convergence theorem. Since also $I_1^\varepsilon \to I_1$, then

$$I_1 = \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma\, d\tau\,. \tag{3.7}$$

Similar,

$$\int_{\Gamma_e(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma \nearrow \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma \le \infty \ \text{ as } \ \varepsilon \to 0\,,$$

for every ω and a.e. τ, where we accept the following convention: $0^{-1} = 0$. By this convergence,

$$I_2^\varepsilon \to \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma\, d\tau \ \text{ as } \ \varepsilon \to 0\,,$$

and

$$I_2 = \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma\, d\tau\,.$$

Now (3.4) implies that

$$\mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma\, d\tau = B_1\, \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma\, d\tau\,, \tag{3.8}$$

where both integrals are finite. Since for each ω the set of critical values τ of ξ has zero measure, then we can arbitrarily re-define $|\nabla\xi|^{-1}$ in critical points of ξ, without changing the integral in the r.h.s.. Below we adopt the following natural convention:

$$\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma = \infty \ \text{ if } \ \tau \ \text{ is a critical value of } \ \xi \tag{3.9}$$

(i.e., if $\Gamma(\tau,\omega)$ is not a finite union of smooth curves). Finally, if τ is a critical value, we define the integral $\int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma \le \infty$ using the limit (3.6). Now the internal integrals in the both sides of (3.8) are defined for all ω and all τ.

Theorem 3.2. *Let $\xi_\nu(t,x)$ be the vorticity of the stationary solution $u_\nu(t,x)$. Then for any $t \ge 0$ and $\nu > 0$ we have*

$$\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla \xi_\nu|\, d\gamma = B_1 \mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla \xi_\nu|^{-1}\, d\gamma,$$

for a.a. $\tau \in \mathbb{R}$. Here $\Gamma(\tau,\omega)$ is defined in (3.5) (where $\xi(x) = \xi_\nu(t,x)$), and we assume (3.9). Moreover, $\int_{\mathbb{R}} (\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla\xi_\nu|\, d\gamma)\, d\tau = \frac{1}{2} M_1$.

Proof: The first assertion follows from the relation (3.8) which holds for any bounded continuous function g. The second assertion follows from (3.7) with $g = 1$ since for $g = 1$ we have $I_1 = \frac{1}{2} M_1$ (see (1.3)). □

4. BOUNDS FOR EXPONENTIAL MOMENTS

As before, we abbreviate $\xi_\nu(t,x) = \xi(x)$. Choosing in (3.3) $g(v) = v^{2m}$, where m is a natural number, and denoting $\xi(x)^{m+1} = w(x)$, we get:

$$\mathbf{E}\int_{\mathbb{T}^2} |\nabla w(x)|^2\, dx = B_1 (m+1)^2\, \mathbf{E}\int_{\mathbb{T}^2} |w(x)|^{\frac{2m}{m+1}}\, dx. \tag{4.1}$$

By the Hölder inequality,

$$\int |w(x)|^{\frac{2m}{m+1}}\, dx \le (2\pi)^{\frac{2}{m+1}} \left(\int w(x)^2\, dx \right)^{\frac{m}{m+1}}. \tag{4.2}$$

By (3.1), $w(x) \in C^1(\mathbb{T}^2)$ for all ω. We wish to estimate the integral of $w(x)^2$ in the r.h.s. of (4.2) by its integral Dirichlet. If m was zero, then $w(x) = \xi(x)$ would be a C^1-smooth function with zero mean-value, and the estimate would follow from the Poincaré inequality. Since $m \ge 1$, then what we need is the 'uniform nonlinear Poincaré inequality', given below. A short and nice proof of this result, suggested by M. Struwe and S. Brandle, is given in Appendix.

Lemma 4.1. *There exists a constant $C \ge 1$ such that for any $k \in \mathbb{N}$ and any C^1-smooth function $u(x)$ on $\mathbb{T}^2$ with zero mean-value we have*

$$\int u^{2k}\, dx \le C \int |\nabla (u^k)|^2\, dx. \tag{4.3}$$

Combining (4.1)–(4.3) we get:

$$\mathbf{E}\int |\nabla w(x)|^2\, dx \le B_1 (m+1)^2 (2\pi)^{\frac{2}{m+1}} C \mathbf{E}\left(\int |\nabla w|^2\, dx \right)^{\frac{m}{m+1}}.$$

Therefore,

$$\mathbf{E}\int |\nabla w(x)|^2\,dx \le C_1^{m+1}(m+1)^{2(m+1)}.$$

Using (4.3) (with $k = m+1$) once again we see that

$$\mathbf{E}\int \xi(x)^{2(m+1)}\,dx \le C_2^{m+1}(m+1)^{2(m+1)}. \tag{4.4}$$

Since ξ is a homogeneous random field, then the l.h.s. equals $(2\pi)^2\mathbf{E}\xi(x)^{2(m+1)}$. Thus, we have proved that

$$(\mathbf{E}|\xi(x)|^j)^{1/j} \le Cj \tag{4.5}$$

for $j = 2r, r = 2, 3, \ldots$. Since for any random variable η the function

$$(0,1] \ni t \to \ln(\mathbf{E}|\eta|^{1/t})^t \le \infty$$

is convex by the Hölder inequality, then (4.5) holds for all $j \in \mathbb{N}$.

For $\sigma > 0$ we have $\mathbf{E}e^{\sigma|\xi(x)|} = \sum_m (\sigma^m/m!)\mathbf{E}|\xi(x)|^m$. As $m! > (m/e)^m$ by the Stirling formula, then (4.5) implies that

$$\mathbf{E}e^{\sigma|\xi(x)|} \le \sum_m (\sigma e C)^m.$$

We have got

Theorem 4.2. *There exists $\sigma > 0$ and $C \ge 1$ such that for any $t \ge 0$, $x \in \mathbb{T}^2$ and $\nu > 0$ we have*

$$\mathbf{E}e^{\sigma|\xi_\nu(t,x)|} \le C. \tag{4.6}$$

Remark 1. A vector field u_ν can be recovered from its vorticity ξ_ν as

$$u_\nu = D\xi_\nu, \qquad D = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)^t (-\Delta)^{-1}, \tag{4.7}$$

where Δ is the Laplacian, operating on functions on $\mathbb{T}^2$ with zero mean-value. The operator D is bounded as a linear map in $L_2(\mathbb{T}^2)$ and as an operator in $L_\infty(T^2)$. Hence, by the interpolation theorem its norm as an operator in $L_p(\mathbb{T}^2)$, $2 \le p \le \infty$, is bounded by a p-independent constant C'. Due to this observation, (4.7) and (4.4),

$$\mathbf{E}\int u_\nu(t,x)^{2(m+1)}\,dx \le C'^{2(m+1)} C_2^{m+1}(m+1)^{2(m+1)}.$$

Therefore

$$\mathbf{E}e^{\sigma_1|u_\nu(t,x)|} \le C_1 \tag{4.8}$$

for some $\sigma_1, C_1 > 0$.

Remark 2. Let us abbreviate $u_\nu(t,x) = u(x)$. Due to (4.7), $\nabla u = \nabla D\xi$. So $\nabla u(x)$ is obtained from $\xi(x)$ by applying a singular integral operator. Hence, $|\nabla u|_{L_p} \le C_p |\xi|_{L_p}$ for $1 < p < \infty$ by the Calderón–Zydmund theorem. For $2 \le p < \infty$ the constant C_p can be chosen $C_p = Cp$, e.g., see [17], section II.6. Due to this estimate with $p = 2(m+1)$ and (4.4) we have

$$\mathbf{E} \int |\nabla u(x)|^{2(m+1)}\, dx \le C_3^{m+1}(m+1)^{4(m+1)}.$$

As before, this inequality implies that $(\mathbf{E}|\nabla u(x)|^{j/2})^{1/j} \le Cj$ for all $j \in \mathbb{N}$ and all x. Therefore,

$$\mathbf{E} e^{\sigma |\nabla u_\nu(t,x)|^{1/2}} \le C_2 \tag{4.9}$$

for any t, x, with some $\sigma_2, C_2 > 0$.

Remark 3. It is shown in [15] that, along sequences $\nu_j \to 0$, the processes $u_{\nu_j}(t,x)$ converge in distribution to limiting stationary processes $u_0(t,x)$ such that a.a. their trajectories satisfy the Euler equation. Using the Fatou lemma it is easy to check that $u_0(t,x)$ and $\xi_0(t,x) = \operatorname{curl} u_0(t,x)$ satisfy (4.5) and (4.6) (as before, first one has to establish analogies of the inequalities (4.4)). Similar, u_0 satisfy (4.8) and (4.9).

5. APPENDIX

5.1. Proof of Lemma (3.1)

Let us consider the space $\mathfrak{A} = [0,1] \times \mathbb{T}^2 \times \Omega$, given the product sigma-algebra and the product-measure $\mathcal{P} = L_t \times L_x \times \mathbf{P}$, where L_t is the Lebesgue measure on [0,1] and L_x is the normalised Lebesgue measure on $\mathbb{T}^2$. Re-defining ζ and ξ_ν on a null-set as at the beginning of Section (3) we achieve that (3.1) hold for all ω.

We set $Q = \{(t,x,\omega) \in \mathfrak{A} \mid \nabla\xi_\nu = 0\}$ (as before, $\nabla = \nabla_x$) and $Q(x) = \{(t,\omega) \mid (t,x,\omega) \in Q\}$, $Q(t,x) = \{\omega \mid (t,x,\omega) \in Q\}$, $Q(x,\omega) = \{t \mid (t,x,\omega) \in Q\}$. Since the random function $\xi_\nu(t,x)$ is stationary in t and x, then $p := \mathbf{P}(Q(t,x))$ is independent of (t,x) and $\mathcal{P}(Q) = p$. We have to prove that $p = 0$. Assume that, on the contrary, $p > 0$.

Let us fix any $x_0 \in \mathbb{T}^2$ and denote by q^ω the set of points of density of $Q(x_0,\omega) \subset [0,1]$. For $t \in [0,1]$ we denote $\pi(t) = \mathbf{P}\{t \in q^\omega\}$. The set $Q^+(x_0) = \{(t,\omega) \mid \omega \in \Omega, t \in q^\omega\}$ is measurable as well as the set $Q(x_0) = \{(t,\omega) \mid \omega \in \Omega, t \in Q(x_0,\omega)\}$ (since the former may be obtained from the latter in a constructive way). Therefore we have

$$\int_0^1 \pi(t)\,dt = \int_0^1 \int_\Omega I_{Q^+(x_0)}\,d\mathbf{P}\,dt = \int_\Omega \left(\int_0^1 I_{q^\omega}\,dt \right) d\mathbf{P}$$
$$= \int_\Omega \left(\int_0^1 I_{Q(x_0,\omega)}\,dt \right) d\mathbf{P} = (L_t \times \mathbf{P})Q(x_0) = p.$$

In particular, there exists $t_0 < 1$ such that $\pi(t_0) > 0$.

Lemma 5.1. *In* $[0, 1]$ *there exists a converging sequence* $t_n \searrow t_0$ *(*$n \geq 1$*) such that*

$$\mathbf{P}\{\omega \mid (t_n, x_0, \omega) \in Q \;\; \forall n \geq 0\} > 0. \tag{5.1}$$

Proof: Since $\pi(t_0) > 0$, then $\delta := \mathbf{P}(S) > 0$, where $S = \{\omega \mid t_0 \in q^\omega\}$. For $0 < \tau \leq 1 - t_0$ we denote $f^\omega(\tau) = \tau^{-1} L_t(q^\omega \cap [t_0, t_0 + \tau]) I_S(\omega)$. Then

$$f^\omega(\tau) \to 1 \quad \text{when} \quad \tau \to 0\,, \quad \text{for each } \omega \in S.$$

Hence, $\mathbf{E} f^\omega(\tau) \to \delta$ as $\tau \to 0$, and for any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\mathbf{E} f^\omega(\tau_\varepsilon) = \int_{t_0}^{t_0+\tau_\varepsilon} \pi_S(\tau)\,\frac{d\tau}{\tau_\varepsilon} \geq \delta(1-\varepsilon), \quad \pi_S(\tau) = \mathbf{P}\{\tau \in q^\omega, \omega \in S\}.$$

Accordingly, for any $\varepsilon > 0$ we can find $\tau'_\varepsilon \in (0, \tau_\varepsilon]$ which satisfies $\pi_S(t_0 + \tau'_\varepsilon) \geq \delta(1-\varepsilon)$. Now we can use induction to construct a sequence $t_n \searrow t_0$ with the property $\pi_S(t_n) \geq \delta(1 - 2^{-n-1})$. Then $\mathbf{P}\{\omega \in S \mid (t_n, x_0, \omega) \in Q \;\; \forall n\} \geq \frac{1}{2}\,\delta$, and (5.1) follows. □

Let us denote

$$\beta(t) = \nabla \operatorname{curl} \zeta(t, x_0), \qquad w(t) = \nabla \xi_\nu(t, x_0) - \sqrt{\nu}\nabla \operatorname{curl} \zeta(t, x_0).$$

By (2.2) and (2.3) $\beta(t)$ is a Brownian motion. By (3.1), $|w(t) - w(t_0)| \leq C^\omega |t - t_0|$ for each ω. So

$$\sqrt{\nu}|\beta(t) - \beta(t_0)| \leq C^\omega |t - t_0| + |\nabla \xi_\nu(t, x_0) - \nabla \xi_\nu(t_0, x_0)|.$$

This inequality and (5.1) imply that the event

$$\{|\beta(t_n) - \beta(t_0)| \leq \nu^{-1/2} C^\omega (t_n - t_0), \quad n = 1, 2, \ldots\} \tag{5.2}$$

has a positive probability. But this is impossible. Indeed, let us consider the event

$$\bigcap_N \bigcup_{n \geq N} \{|\beta(t_n) - \beta(t_0)| \geq \sqrt{t_n - t_0}\}. \tag{5.3}$$

We claim that its probability is one. Since (5.2) and (5.3) do not intersect, then the probability of (5.2) must be zero.

It remains to verify the Borel–Cantelli like claim we have made. It suffice to prove that the probability one has the event, obtained from (5.3) by replacing the sequence $\{t_n\}$ by any its subsequence $\{t'_n\}$. To do this it suffices to check that

$$\mathbf{P}\Big\{ \cap_{n\ge k} \big\{|\beta(t'_n)-\beta(t_0)| \le \sqrt{t'_n - t_0}\big\}\Big\} = 0\,, \tag{5.4}$$

for any $k \ge 1$. The l.h.s. is bounded by

$$\mathbf{E}\prod_{n=k}^{K} I_{\{|\beta(t'_n)-\beta(t_0)|\le\sqrt{t'_n-t_0}\}}\,, \tag{5.5}$$

for any fixed $K > k$. If $t_{n+1} - t_0 \ll t_n - t_0$, then

$$\mathbf{E}\left(I_{\{|\beta(t'_n)-\beta(t_0)|\le\sqrt{t'_n-t_0}\}} \mid \beta(t'_{n+1})\right) \le \mathbf{P}\{|\beta(t'_n)-\beta(t_0)| \le 2\sqrt{t'_n - t_0}\} =: c\,,$$

for each $\beta(t'_{n+1})$, satisfying $|\beta(t'_{n+1}) - \beta_{t_0}| \le \sqrt{t'_{n+1} - t_0}$. Now, choosing for $\{t'_n\}$ a subsequence, converging to t_0 fast enough and using the Markov property, we see that (5.5) $\le c^{K-k+1}$. Since $c < 1$ and K is any number $> k$, then (5.4) follows.

5.2. Proof of Lemma (4.1)

Assume that the lemma's assertion is wrong. Then there exists a sequence of functions $\{u_l\} \subset C^1(\mathbb{T}^2)$ with zero mean-value, and a sequence of integers $\{p_l \ge 1\}$ such that for $v_l(x) = \operatorname{sgn} u_l(x)\,|u_l(x)|^{p_l}$ we have

$$\int v_l^2\,dx \ge l \int |\nabla v_l|^2\,dx. \tag{5.6}$$

Without loss of generality we may assume that

$$\int v_l^2\,dx = (2\pi)^2 \quad \forall\, l. \tag{5.7}$$

Due to (5.6) and (5.7) the sequence $\{v_l\}$ is bounded in $H^1(\mathbb{T}^2)$. So it contains a converging subsequence $\{v_{l_j}\}$:

$$v_{l_j} \to v \;\text{ weakly in }\; H^1(T^2) \;\text{ and }\; v_{l_j} \to v \;\text{ strongly in }\; L_2(T^2). \tag{5.8}$$

By (5.7), $\|v\|_{L_2} = 2\pi$. Due to (5.6), $\|\nabla v_l\|_{L_2} \le 2\pi/\sqrt{l}$. Therefore $\|\nabla v\|_{L_2} \le \liminf \|\nabla v_{l_j}\|_{L_2} = 0$, and $v(x) \equiv 1$.

Now let us consider the function

$$\phi(p,u) = \left|\frac{|u|^p \operatorname{sgn} u - 1}{u-1}\right|, \quad p \ge 1,\; u \ne 1.$$

We claim that

$$\phi(p,u) > 1/2 \quad \forall\, p \geq 1,\ \forall\, u \neq 1. \tag{5.9}$$

Indeed, note first that $\phi(1,u) \equiv 1$. It is easy to check that as a function of $p \geq 1$, $\phi(p,u)$ increases with p if $|u| > 1$ and if $u \in (0,1)$, so for such u the estimate holds. It remains to consider the case $u \in [-1,0]$. But now

$$\phi = \frac{|u|^p + 1}{|u| + 1} > \frac{1}{2}\,.$$

By (5.9), $|u_l(x) - 1| \leq 2|v_l(x) - 1|$. Due to this inequality and (5.8) (where $v \equiv 1$), we have that $u_{l_j} \to 1$ in L_1. This contradicts the normalisation $\int u_l\, dx = 0$.

ACKNOWLEDGMENTS

I thank N. Nadirashvili for discussions, M. Struwe and S. Brandle—for suggesting the proof of Lemma (4.1), presented in this work, and a referee—for drawing my attention to the works [4, 5, 2]. The paper was written during my staying at FIM (ETH-Zürich), and I thank the institute for hospitality. This work is a part of a research, supported by EPSRC, grant S68712/01.

REFERENCES

1. J. Bricmont, A. Kupiainen and R. Lefevere, Exponential mixing for the 2D stochastic Navier–Stokes dynamics, *Comm. Math. Phys.* **230** no. 1, 87–132 (2002).
2. P. Constantin and Ch. Doering, Heat transfer in convective turbulence, *Nonlinearity* **9**, 1049–1060 (1996).
3. P. Constantin and C. Foias, Navier-Stokes Equations, University of Chicago Press, Chicago (1988).
4. P. Constantin, Navier–Stokes equations and area of interfaces, *Comm. Math. Phys.* **129**, 241–266 (1990).
5. P. Constantin and I. Procaccia, The geometry of turbulent advection: sharp estimates for the dimension of level sets, *Nonlinearity* **7**, 1045–1054 (1994).
6. G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge (1992).
7. W. E, J. C. Mattingly and Ya. G. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, *Comm. Math. Phys.* **224**, 83–106 (2001).
8. F. Flandoli, Dissipativity and invariant measures for stochastic Navier-Stokes equations, *NoDEA* **1**, 403–423 (1994).
9. M. Hairer and J. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, Preprint (2004).
10. S. B. Kuksin and O. Penrose, A family of balance relations for the two-dimensional Navier–stokes eqautions with random forcing, *J. Stat. Physics* **118**, 437–449 (2005).
11. S. B. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, *Comm. Math. Phys.* **213**, 291–330 (2000).
12. S. B. Kuksin and A. Shirikyan, Coupling approach to white-forced nonlinear PDE's, *J. Math. Pures Appl.* **81**, 567–602 (2002).

13. S. B. Kuksin and A. Shirikyan, Some limiting properties of randomly forced 2D Navier–Stokes equations, *Proceedings A of the Royal Society of Edinburgh* **133**, 875–891 (2003).
14. S. B. Kuksin, Ergodic theorems for 2D statistical hydrodynamics, *Rev. Math. Phys.* **14**, 585–600 (2002).
15. S. B. Kuksin, The Eulerian limit for 2D statistical hydrodynamics, *J. Stat. Physics* **115**, 469–492 (2004).
16. A. Shirikyan, Law of large numbers and central limit theorem for randomly forced PDE's, *Ergodic Theory and Related Fields*, to appear (2005).
17. E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton (1970).
18. M. I. Vishik and A. V. Fursikov, Mathematical Problems in Statistical Hydromechanics, Kluwer, Dordrecht (1988).